Problem 1.65

(a) Check Stokes' theorem for the vector function

$$\mathbf{A} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2 + y^2},$$

using a circle of radius R in the xy plane. Diagnose the problem, and fix it by correcting the curl of $\mathbf{A}$. Use Cartesian coordinates.

(b) Convert $\mathbf{A}$ to cylindrical coordinates, and repeat part (a), this time doing everything in terms of s, ϕ, and z.

Solution

Part (a)

According to Stokes's theorem,

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{A} \cdot d\mathbf{l}$$

$$\iint_{x^2+y^2 \le R^2} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} \cdot (\hat{\mathbf{z}}\, dx\, dy) = \oint_{x^2+y^2=R^2} \mathbf{A} \cdot d\mathbf{l}$$

$$\iint_{x^2+y^2 \le R^2} \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] \hat{\mathbf{z}} \cdot (\hat{\mathbf{z}}\, dx\, dy) = \oint_{(R\cos t)^2+(R\sin t)^2=R^2} \mathbf{A}[\mathbf{l}(t)] \cdot \mathbf{l}'(t)\, dt$$

$$\iint_{x^2+y^2 \le R^2} \left[\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] dx\, dy = \int_0^{2\pi} \frac{-(R\sin t)\hat{\mathbf{x}} + (R\cos t)\hat{\mathbf{y}}}{(R\cos t)^2 + (R\sin t)^2} \cdot \langle -R\sin t, R\cos t, 0\rangle\, dt$$

$$\iint_{x^2+y^2 \le R^2} (0)\, dx\, dy = \int_0^{2\pi} \frac{-\sin t\,\hat{\mathbf{x}} + \cos t\,\hat{\mathbf{y}}}{1} \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$0 = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} dt$$

$$\neq 2\pi.$$

The problem with the curl of $\mathbf{A}$ is that it's implicitly assuming that $x^2 + y^2 \neq 0$.

Change it to

$$\nabla \times \mathbf{A} = 2\pi\delta(x)\delta(y)\hat{\mathbf{z}}$$

so that Stokes's theorem is satisfied.

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \iint_{x^2+y^2\leq R^2} 2\pi\delta(x)\delta(y)\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}}\, dx\, dy) = 2\pi \int_{-R}^{R} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \delta(x)\delta(y)\, dx\, dy = 2\pi \int_{-R}^{R} \delta(y)\, dy = 2\pi$$

The point (0,0) lies within the circle of radius R, so the double integral evaluates to 1.

Part (b)

Convert $\mathbf{A}$ to cylindrical coordinates.

$$\begin{aligned}\mathbf{A} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2+y^2} &= \frac{-s\sin\phi(\cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}) + s\cos\phi(\sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}})}{(s\cos\phi)^2 + (s\sin\phi)^2} \\ &= \frac{s\sin^2\phi\,\hat{\boldsymbol{\phi}} + s\cos^2\phi\,\hat{\boldsymbol{\phi}}}{s^2} \\ &= \frac{\hat{\boldsymbol{\phi}}}{s}\end{aligned}$$

According to Stokes's theorem,

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{A} \cdot d\mathbf{l}$$

$$\iint_{s\leq R} \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi)\right]\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}}\, s\, ds\, d\phi) = \oint_{s=R} \left(\frac{\hat{\boldsymbol{\phi}}}{s}\right) \cdot (s\, d\phi\, \hat{\boldsymbol{\phi}})$$

$$\int_0^{2\pi} \int_0^R \left[\frac{d}{ds}(1)\right] ds\, d\phi = \int_0^{2\pi} d\phi$$

$$\int_0^{2\pi} \int_0^R (0)\, ds\, d\phi = 2\pi$$

$$0 \neq 2\pi.$$

The problem with the curl of $\mathbf{A}$ is that it's implicitly assuming that $s \neq 0$. Change it to

$$\nabla \times \mathbf{A} = \frac{\delta(s)}{s}\hat{\mathbf{z}}$$

so that Stokes's theorem is satisfied.

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \iint_{s\leq R} \frac{\delta(s)}{s}\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}}\, s\, ds\, d\phi) = \int_0^{2\pi} \int_0^R \delta(s)\, ds\, d\phi = \int_0^{2\pi} (1)\, d\phi = 2\pi$$